Week 6 Linear Functional

Defn Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. X be vector space$/\mathbb{F}$

A linear functional is a linear operator

$$f : \mathcal{D}(f) \longrightarrow \mathbb{F}$$

where $\mathcal{D}(f) \subseteq X$ is a vector subspace

Defn If X is a normed space, then a linear functional is said to bounded if $\exists\, c > 0$ s.t.

$$|f(x)| \leq c\|x\| \quad \forall x \in X$$

Define $\|f\| = \sup\limits_{x \neq 0} \dfrac{|f(x)|}{\|x\|}$

Notation

$X^* =$ algebraic dual space of X (vector space X)

$=$ the set of all linear functionals on X

$X' =$ dual space of X ← (normed space X)

$=$ the set of all bounded linear functionals on X

Note ① X^* is a vector space $/\mathbb{F}$

② $X' = B(X, \mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is a Banach space (complete normed space)

(Thm 2.10-2 + $\mathbb{R}$ and $\mathbb{C}$ are complete)

③ If $\dim X < \infty$, then $X' = X^*$

(Thm 2.7-8)

Recall: Thm 2.10-2

Let Y be a Banach space

X be a normed space

Then $B(X,Y)$ is a Banach space

Idea of Pf

Let $\{T_n\}$ be a Cauchy sequence in $B(X,Y)$

Then $\forall x \in X$,

$\{T_n(x)\}$ is a Cauchy sequence in Y

Complete

$\Rightarrow \lim_{n\to\infty} T_n(x)$ exists

Let $T(x) = \lim_{n\to\infty} T_n(x)$

Need to show: T is linear, bounded

and $T_n \to T$ in $B(X,Y)$

eg $(\mathbb{R}^2)^* = (\mathbb{R}^2)'$ $x \in \mathbb{R}^2$, $\|x\| = \sqrt{x_1^2 + x_2^2}$

Let $f \in (\mathbb{R}^2)'$, ie f is a bounded linear functional on $\mathbb{R}^2$

Let $\beta = \{e_1, e_2\}$ be basis for $\mathbb{R}^2$ $e_1 = [1,0]$, $e_2 = [0,1]$

Consider $S : (\mathbb{R}^2)' \to \mathbb{R}^2$ defind by

$S(f) = (f(e_1), f(e_2))$

Want to show that S is linear and bijective

i. $S(f+g) = ((f+g)(e_1), (f+g)(e_2))$

$= (f(e_1) + g(e_1), f(e_2) + g(e_2))$

$= (f(e_1), f(e_2)) + (g(e_1), g(e_2))$

$= S(f) + S(g)$

ii $S(\alpha f) = (\alpha f(e_1), \alpha f(e_2))$

$= \alpha (f(e_1), f(e_2))$

$= \alpha S(f)$

$\Rightarrow$ S is linear

iii Suppose $S(f) = (0,0)$

$\Rightarrow (f(e_1), f(e_2)) = (0,0)$

$\Rightarrow f(e_1) = f(e_2) = 0$

$f(x,y) = f(x(1,0) + y(0,1))$

$= x f(1,0) + y f(0,1)$

$= x f(e_1) + y f(e_2)$

$= 0 + 0 = 0$

$\Rightarrow f =$ zero linear functional

S is linear $\Rightarrow$ S is injective

iv. Let $(a,b) \in \mathbb{R}^2$

Define $f(x,y) = ax + by$

Then $f \in (\mathbb{R}^2)'$ and

$S(f) = (f(e_1), f(e_2)) = (a,b)$

$\Rightarrow S$ is surjective

Rmk $f \in (\mathbb{R}^2)'$, $f(x,y) = x f(e_1) + y f(e_2)$

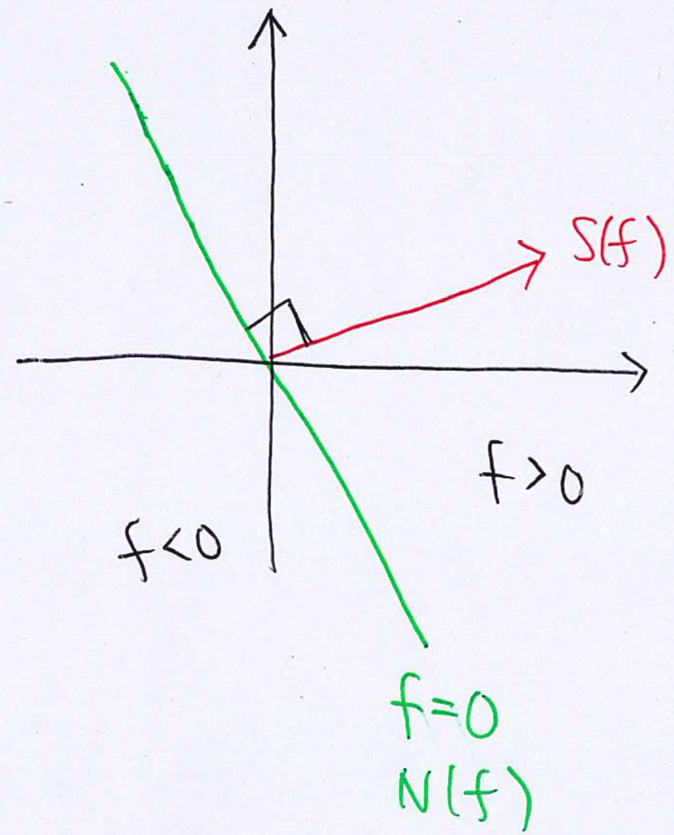
$= (f(e_1), f(e_2)) \cdot (x,y)$

$= S(f) \cdot (x,y)$

Every bounded linear functional $f : \mathbb{R}^2 \to \mathbb{R}$

is given by taking dot product with $S(f)$

Picture



Defn (Isometry)

Let X, Y be normed space

A map $T: X \to Y$ is called an isometry if

1. T is linear
2. T is bijective
3. $\|T(x)\|_Y = \|x\|_X \quad \forall x \in X$

Last example: $S: (\mathbb{R}^2)' \longrightarrow \mathbb{R}^2$ satisfies 1, 2

Indeed S also satisfies 3 (HW: Cauchy-Schwarz)

$\Rightarrow$ S is an isometry

$X \cong Y$

If such a T exists, then we say X and Y are isometric

Dual space of ℓ^p (Two cases: $p=1$, $1<p<\infty$)

$$\boxed{(\ell^1)' \cong \ell^\infty}$$

To prove they are isometric, we will construct isometry between them

Step 1 Define

$S : (\ell^1)' \to \ell^\infty$ and $T : \ell^\infty \to (\ell^1)'$:

For $f \in (\ell^1)'$, let

$$\boxed{S(f) = (f(e_n))_n = (f(e_1), f(e_2), f(e_3), \cdots)}$$

n-th term of a sequence

Q: Is it in ℓ^∞?

Check: Note $|f(e_n)| \le \|f\| \|e_n\| = \|f\|$

$\Rightarrow \|S(f)\|_\infty \le \|f\|$ and $S(f) \in \ell^\infty$

Also, for $\vec{y} = (y_1, y_2, y_3, \cdots) \in \ell^\infty$

let $\boxed{T(\vec{y}) : \ell^1 \to \mathbb{R}}$ defined by

$$\boxed{T(\vec{y})(\vec{x}) = \sum_{i=1}^\infty x_i y_i \quad \text{for } \vec{x} = (x_1, x_2, \cdots) \in \ell^1}$$

Worry: Is $T : \ell^\infty \to (\ell^1)'$ well-defined?

ⓐ $\sum_{i=1}^\infty x_i y_i$ convergent?

ⓑ $T(\vec{y}) \in (\ell^1)'$? i.e. linear? bounded?

For ⓐ, note

$$\sum_{i=1}^\infty |x_i y_i| = \sum_{i=1}^\infty |x_i||y_i|$$

$$\le \sum_{i=1}^\infty |x_i| \|y\|_\infty = \|y\|_\infty \sum_{i=1}^\infty |x_i| = \|y\|_\infty \|x\|_1 < \infty$$

$\Rightarrow \sum_{i=1}^\infty x_i y_i$ is convergent (i.e. $\lim_{n\to\infty} \sum_{i=1}^n x_i y_i$ exists)

For b

bi $T(\vec{y})(\vec{x}_1+\vec{x}_2)$

$$= \sum_{i=1}^{\infty}(x_{1,i}+x_{2,i})y_i$$

$$= \sum_{i=1}^{\infty} x_{1,i}y_i + \sum_{i=1}^{\infty} x_{2,i}y_i$$

$$= T(\vec{y})(\vec{x}_1) + T(\vec{y})(\vec{x}_2)$$

Similarly, $T(\vec{y})(\alpha\vec{x})$

$$= \alpha\, T(\vec{y})(\vec{x})$$

bii $|T(\vec{y})(\vec{x})| = \left|\sum_{i=1}^{\infty} x_i y_i\right|$

$$\leq \sum_{i=1}^{\infty} |x_i y_i|$$

$$\leq \|y\|_\infty \|\vec{x}\|_1$$

$\Rightarrow$ $T(\vec{y})$ is bounded and $\|T(\vec{y})\| \overset{(*)}{\leq} \|y\|_\infty$

$\therefore$ T is well-defined

Step 2 Check S, T are linear (Exercise)

e.g. $S(f+g) = S(f) + S(g)$?

$S(\alpha f) = \alpha S(f)$?

Step 3 Check T and S are inverses

$$\underbrace{T(S(f))}_{(\ell')'}(\vec{x}) = \sum_{i=1}^{\infty} x_i\, S(f)_i$$

$$= \sum_{i=1}^{\infty} x_i f(e_i) \overset{?}{=} f\left(\sum_{i=1}^{\infty} x_i e_i\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} x_i f(e_i)$$

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$$= \lim_{n\to\infty} f\left(\sum_{i=1}^{n} x_i e_i\right) \quad (f \text{ is linear})$$

$$= f\left(\lim_{n\to\infty} \sum_{i=1}^{n} x_i e_i\right) \quad \left(\begin{array}{l} f \text{ is bounded} \\ \Rightarrow f \text{ is continuous}\end{array}\right)$$

$$= f\left(\sum_{i=1}^{\infty} x_i e_i\right)$$

$$= f(\vec{x}) \quad \text{for any } \vec{x} \in \ell^1$$

$$\Rightarrow T(S(f)) = f$$

$$e_{n,i} = \begin{cases} 1 & \text{if } n = i \\ 0 & \text{if } n \neq i \end{cases}$$

Similarly:

$$S(T(\vec{y})) = \left(T(\vec{y})(e_n)\right)_n$$

$$= \left(\sum_{i=1}^{\infty} e_{n,i}\, y_i\right)_n$$

$$= (y_n)_n = \vec{y}$$

$$\Rightarrow S(T(\vec{y})) = \vec{y}$$

$\therefore$ S, T are inverses of each other

<u>Step 4</u> Show $\|S(f)\| = \|f\|$

$$\|S(f)\|_\infty = \sup_n |f(e_n)|$$

$$\leq \sup_n \|f\| \|e_n\|_1$$

$$= \sup_n \|f\| = \|f\|$$

Also, $\|f\| = \|T(S(f))\| \leq \|S(f)\|_\infty$

$\because \|T(\vec{y})\| \leq \|y\|_\infty$ (*) on P6 for any $\vec{y} \in \ell^\infty$

$\boxed{\text{(1)-(4)} \Rightarrow S \text{ is an isometry } (\ell^1)' \cong \ell^\infty}$

If $1 < p < \infty$, let $q \in \mathbb{R}$ st.

$$\frac{1}{p} + \frac{1}{q} = 1$$

Then $\boxed{(\ell^p)' \cong \ell^q}$

Pf is similar but more technical

Outline of steps:

1. Define $S : (\ell^p)' \to \ell^q$, $T : \ell^q \to (\ell^p)'$
2. Prove S, T are well-defined
3. Prove S, T are linear
4. Prove $S \circ T$ and $T \circ S$ are identities
5. $\|S(f)\| = \|f\| \quad \forall f \in (\ell^p)'$

Rmk Defn of S, T in 1

For $f \in (\ell^p)'$, define

$$S(f) = (f(e_n))_n$$

For $\vec{y} \in \ell^q$, define

$$T(\vec{y}) : \ell^p \to \mathbb{R} \text{ by}$$

$$T(\vec{y})(\vec{x}) = \sum_{i=1}^{\infty} x_i y_i$$

exactly the same as in $p=1$

Rmk 2

$$\boxed{(\ell^\infty)' \ncong \ell^1}$$

Bigger

Hahn-Banach Theorem for real normed space (Thm 4.3-2)

Let X be a real normed space and

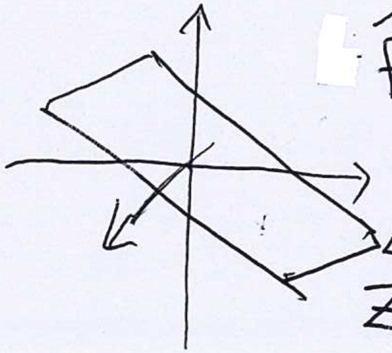
$Z \subseteq X$ be a subspace

Suppose f is a bounded linear functional on Z

i.e. $f : Z \to \mathbb{R}$ is linear, bounded

Then $\exists$ linear extension $\hat{f} : X \to \mathbb{R}$ such that

$$\|\hat{f}\| = \|f\|$$



$\hat{f}$ is defnd on X

f is defnd on Z

Z

Rmk

There are also complex version and a more general version (4.3-1)

Proposition (Extension by one dimension)

Suppose $y \in X \setminus Z$ and let

$$Y = Z \oplus \operatorname{span}(y)$$

i.e. $Y = \{ z + \alpha y : z \in Z, \alpha \in \mathbb{R} \}$

Then $\exists$ a linear extension $\hat{f} : Y \to \mathbb{R}$ such that

$$\|\hat{f}\| = \|f\|$$

Pf Note that if $\tilde{f}$ is linear, then

$$\tilde{f}(z+\alpha y) = \tilde{f}(z) + \alpha \tilde{f}(y)$$

$$= f(z) + \alpha \tilde{f}(y)$$

$\therefore$ Define $\tilde{f}$ $\Leftrightarrow$ Define $\underbrace{\tilde{f}(y)}_{\text{We will do this}}$

Let $M = \|f\|$. Let $z_1, z_2 \in Z$

$$f(z_1) - f(z_2) = f(z_1 - z_2)$$

$$\leq M \|z_1 - z_2\|$$

$$= M \|(z_1 + y) - (z_2 + y)\|$$

$$\leq M (\|z_1 + y\| + \|z_2 + y\|)$$

$$\underbrace{-M\|z_2+y\| - f(z_2)}_{\text{indept of } z_1} \leq \underbrace{M\|z_1+y\| - f(z_1)}_{\text{indept of } z_2}$$

$\Rightarrow \exists\, c \in \mathbb{R}$ s.t. for any z_1, z_2

$$-M\|z_2+y\| - f(z_2) \leq c \leq M\|z_1+y\| - f(z_1)$$

In particular, if $z_1 = z_2$, then

$$-M\|z_1+y\| - f(z_1) \leq c \leq M\|z_1+y\| - f(z_1) \quad (*)$$

Define $\tilde{f}(y) = c$

Note $(*)$ $\Rightarrow$ For any $z_1 \in Z$

$$-M\|z_1+y\| \leq \underset{\displaystyle f(z_1) + \tilde{f}(y)}{\overset{\shortparallel}{f(z_1) + c}} \leq M\|z_1+y\|$$

$$\Rightarrow \left|f(z_1)+\hat{f}(y)\right| \le M\|z_1+y\| \quad (**)$$

Want to prove

$$\left|\tilde{f}(z+\alpha y)\right| \le M\|z+\alpha y\|$$

Case 1 : $\alpha=0$ $\quad \hat{f}(z+\alpha y) = f(z)$

$$|f(z)| \le M\|z\| \quad \text{since } \|f\|=M$$

Case 2 : $\alpha \neq 0$

Let $z_1 = \frac{z}{\alpha}$ $\quad (**) \Rightarrow$

$$\left|f\left(\tfrac{z}{\alpha}\right)+\hat{f}(y)\right| \le M\left\|\tfrac{z}{\alpha}+y\right\|$$

$$\left|f(z)+\alpha\hat{f}(y)\right| \le M\|z+\alpha y\|$$

$$\Rightarrow \left|\tilde{f}(z+\alpha y)\right| \le M\|z+\alpha y\| \quad \blacksquare$$

Pf of 4.3-2

Case 1 If $X = Z \oplus \text{span}\{y_1, y_2, \cdots, y_n\}$

where $y_{k+1} \notin \underbrace{Z \oplus \text{span}\{y_1, y_2, \cdots y_k\}}_{\text{call it } Y_k}$

Then

$$Z = Y_0 \le Y_1 \le Y_2 \le \cdots \le Y_n = X$$

dim is differed by 1 in each step

f defined on Y_0

Prop $\Rightarrow$ Extend $\hat{f}_1$ to Y_1

Prop again $\Rightarrow$ Extend $\hat{f}_2$ to Y_2

Repeat n times $\Rightarrow$ Extend f to $\tilde{f}$ on X

Case 2

If $X = Z \oplus \text{span}\underbrace{\{y_1, y_2, y_3, \cdots\}}_{\text{countable, infinite}}$

such that

$$y_{k+1} \notin Y_k = Z \oplus \text{span}\{y_1, y_2, \cdots, y_k\}$$

Repeated application of prop

$\Rightarrow$ f can be extended to

$$\tilde{f}_k : Y_k \to \mathbb{R}$$

But $X = \bigcup_{k=1}^{\infty} Y_k$

Any vector $x \in X$ belong to Y_k for some large enough k

$\Rightarrow$ $\hat{f}$ can be defined from $\hat{f}_k$

Case 3 Otherwise

X is "much bigger" than Z

Need $\underbrace{\text{"Zorn's lemma"}}$ (4.1)

An axiom in set theory

(i.e. Big and important assumption)